

# HERMITE NORMAL FORMS AND $\delta$ -VECTOR

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**ABSTRACT.** Let  $\delta(\mathcal{P}) = (\delta_0, \delta_1, \dots, \delta_d)$  be the  $\delta$ -vector of an integral polytope  $\mathcal{P} \subset \mathbb{R}^N$  of dimension  $d$ . Following the previous work of characterizing the  $\delta$ -vectors with  $\sum_{i=0}^d \delta_i \leq 3$ , the possible  $\delta$ -vectors with  $\sum_{i=0}^d \delta_i = 4$  will be classified. And each possible  $\delta$ -vectors can be obtained by simplices. We get this result by studying the problem of classifying the possible integral simplices with a given  $\delta$ -vector  $(\delta_0, \delta_1, \dots, \delta_d)$ , where  $\sum_{i=0}^d \delta_i \leq 4$ , by means of Hermite normal forms of square matrices.

## 1. INTRODUCTION

**1.1. Background for  $\delta$ -vectors.** Let  $\mathcal{P} \subset \mathbb{R}^N$  be an integral polytope of dimension  $d$  and  $\partial\mathcal{P}$  its boundary. Define the numerical functions  $i(\mathcal{P}, n)$  and  $i^*(\mathcal{P}, n)$  by setting

$$i(\mathcal{P}, n) = |n\mathcal{P} \cap \mathbb{Z}^N|, \quad i^*(\mathcal{P}, n) = |n(\mathcal{P} - \partial\mathcal{P}) \cap \mathbb{Z}^N|.$$

Here  $n\mathcal{P} = \{n\alpha : \alpha \in \mathcal{P}\}$  and  $|X|$  is the cardinality of a finite set  $X$ . The systematic study of  $i(\mathcal{P}, n)$  and  $i^*(\mathcal{P}, n)$  originated in Ehrhart [1] around 1955, who established the following fundamental properties:

- (0.1)  $i(\mathcal{P}, n)$  is a polynomial in  $n$  of degree  $d$ ;
- (0.2)  $i(\mathcal{P}, 0) = 1$ ;
- (0.3) (reciprocity law)  $i^*(\mathcal{P}, n) = (-1)^d i(\mathcal{P}, -n)$  for every integer  $n > 0$ .

We say that  $i(\mathcal{P}, n)$  is the *Ehrhart polynomial* of  $\mathcal{P}$ . An introduction to Ehrhart polynomials is discussed in [8, pp. 235–241] and [2, Part II].

We define the sequence  $\delta_0, \delta_1, \delta_2, \dots$  of integers by the formula

$$(1) \quad (1 - \lambda)^{d+1} \left( 1 + \sum_{n=1}^{\infty} i(\mathcal{P}, n) \lambda^n \right) = \sum_{i=0}^{\infty} \delta_i \lambda^i.$$

In particular,  $\delta_0 = 1$  and  $\delta_1 = |\mathcal{P} \cap \mathbb{Z}^N| - (d+1)$ . Thus, if  $\delta_1 = 0$ , then  $\mathcal{P}$  is a simplex. The above facts (0.1) and (0.2) together with a well-known result on generating function ([8, Corollary 4.3.1]) guarantee that  $\delta_i = 0$  for every  $i > d$ . We say that the sequence

$$\delta(\mathcal{P}) = (\delta_0, \delta_1, \dots, \delta_d)$$

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which appears in Eq. (1) is the  $\delta$ -vector of  $\mathcal{P}$  and the polynomial

$$\delta_{\mathcal{P}}(t) = \delta_0 + \delta_1 t + \cdots + \delta_d t^d$$

which also appears in Eq. (1) is the  $\delta$ -polynomial of  $\mathcal{P}$ .

It follows from the reciprocity law (0.3) that

$$(1 - \lambda)^{d+1} \left( \sum_{n=1}^{\infty} i^*(\mathcal{P}, n) \lambda^n \right) = \sum_{i=0}^d \delta_{d-i} \lambda^{i+1}.$$

In particular,  $\delta_d = |(\mathcal{P} - \partial\mathcal{P}) \cap \mathbb{Z}^N|$ . Each  $\delta_i$  is nonnegative ([9]). If  $\delta_d \neq 0$ , then  $\delta_1 \leq \delta_i$  for every  $1 \leq i < d$  ([3]).

Let  $s = \max\{i : \delta_i \neq 0\}$ . Stanley [10] shows that

$$(2) \quad \delta_0 + \delta_1 + \cdots + \delta_i \leq \delta_s + \delta_{s-1} + \cdots + \delta_{s-i}, \quad 0 \leq i \leq \lfloor s/2 \rfloor$$

by using Cohen–Macaulay rings. The inequalities

$$(3) \quad \delta_{d-1} + \delta_{d-2} + \cdots + \delta_{d-i} \leq \delta_2 + \delta_3 + \cdots + \delta_i + \delta_{i+1}, \quad 1 \leq i \leq \lfloor (d-1)/2 \rfloor$$

appear in [3, Remark (1.4)].

**1.2. Main result: characterization of  $\delta$ -vectors with  $\sum_{i=0}^d \delta_i = 4$ .** One of the most fundamental problems of enumerative combinatorics is to find a combinatorial characterization of all vectors that can be realized as the  $\delta$ -vector of some integral polytope. For example, restrictions like  $\delta_0 = 1$ ,  $\delta_i \geq 0$ , (2) and (3) are necessary conditions for a vector to be a  $\delta$ -vector of some integral polytope.

On the one hand, the complete classification of the  $\delta$ -vectors for dimension 2 is given essentially by Scott [7], while the case where the dimension is greater than or equal to 3 is presumably unknown. In [4], on the other hand, the possible  $\delta$ -vectors with  $\sum_{i=0}^d \delta_i \leq 3$  are completely classified by the inequalities (2) and (3).

**Theorem 1.1.** [4, Theorem 0.1] *Let  $d \geq 3$ . Given a sequence  $(\delta_0, \delta_1, \dots, \delta_d)$  of nonnegative integers, where  $\delta_0 = 1$  and  $\delta_1 \geq \delta_d$ , which satisfies  $\sum_{i=0}^d \delta_i \leq 3$ , there exists an integral polytope  $\mathcal{P} \subset \mathbb{R}^d$  of dimension  $d$  whose  $\delta$ -vector coincides with  $(\delta_0, \delta_1, \dots, \delta_d)$  if and only if  $(\delta_0, \delta_1, \dots, \delta_d)$  satisfies all inequalities (2) and (3).*

However, this is not true for  $\sum_{i=0}^d \delta_i = 4$  ([4, Example 1.2]). In this paper, we will give the complete classification of the possible  $\delta$ -vectors with  $\sum_{i=0}^d \delta_i = 4$  (see Theorem 5.1). Moreover, similarly to  $\sum_{i=0}^d \delta_i \leq 3$ , it turns out that all the possible  $\delta$ -vectors with  $\sum_{i=0}^d \delta_i = 4$  can be chosen to be integral simplices, although this does not hold when  $\sum_{i=0}^d \delta_i = 5$  (Remark 5.3).

### 1.3. Approach: a classification of integral simplices with a given $\delta$ -vector.

Let  $\mathbb{Z}^{d \times d}$  denote the set of  $d \times d$  integral matrices. Recall that a matrix  $A \in \mathbb{Z}^{d \times d}$  is *unimodular* if  $\det(A) = \pm 1$ . Given integral polytopes  $\mathcal{P}$  and  $\mathcal{Q}$  in  $\mathbb{R}^d$  of dimension  $d$ , we say that  $\mathcal{P}$  and  $\mathcal{Q}$  are *unimodularly equivalent* if there exists a unimodular matrix  $U \in \mathbb{Z}^{d \times d}$  and an integral vector  $w$ , such that  $\mathcal{Q} = f_U(\mathcal{P}) + w$ , where  $f_U$  is the linear transformation in  $\mathbb{R}^d$  defined by  $U$ , i.e.,  $f_U(\mathbf{v}) = \mathbf{v}U$  for all  $\mathbf{v} \in \mathbb{R}^d$ . Clearly, if  $\mathcal{P}$  and  $\mathcal{Q}$  are unimodularly equivalent, then  $\delta(\mathcal{P}) = \delta(\mathcal{Q})$ . Conversely, given a vector  $v \in \mathbb{Z}_{\geq 0}^{d+1}$ , it is natural to ask what are all the integral polytopes  $\mathcal{P}$  under unimodular equivalence, such that  $\delta(\mathcal{P}) = v$ .

In this paper, we will focus on this problem for simplices with one vertex at the origin. In addition, we do not allow any shifts in the equivalence, i.e.,  $d$ -dimensional integral polytopes  $\mathcal{P}$  and  $\mathcal{Q}$  are equivalent if there exists a unimodular matrix  $U$ , such that  $\mathcal{Q} = f_U(\mathcal{P})$ . By considering the  $\delta$ -vectors of all the integral simplices up to this equivalence whose normalized volumes are 4, we will get our main result Theorem 5.1.

For discussing the representative under equivalence of the integral simplices with one vertex at the origin, we consider Hermite normal forms.

Let  $\mathcal{P}$  be an integral simplex in  $\mathbb{R}^d$  of dimension  $d$  with the vertices  $\mathbf{0}, \mathbf{v}_1, \dots, \mathbf{v}_d$ . Define  $M(\mathcal{P}) \in \mathbb{Z}^{d \times d}$  to be the matrix with the row vectors  $\mathbf{v}_1, \dots, \mathbf{v}_d$ . Then we have the following connection between the matrix  $M(\mathcal{P})$  and the  $\delta$ -vector of  $\mathcal{P}$ :  $|\det(M(\mathcal{P}))| = \sum_{i \geq 0} \delta_i$ . In this setting,  $\mathcal{P}$  and  $\mathcal{P}'$  are equivalent if and only if  $M(\mathcal{P})$  and  $M(\mathcal{P}')$  have the same Hermite normal form, where the *Hermite normal form* of a nonsingular integral square matrix  $B$  is the unique nonnegative lower triangular matrix  $A = (a_{ij}) \in \mathbb{Z}_{\geq 0}^{d \times d}$  such that  $A = BU$  for some unimodular matrix  $U \in \mathbb{Z}^{d \times d}$  and  $0 \leq a_{ij} < a_{ii}$  for all  $1 \leq j < i$ , (see [6, Chapter 4]). In other words, we can pick the Hermite normal form as the representative in each equivalence class and study the following

**Problem 1.2.** Given a vector  $v \in \mathbb{Z}_{\geq 0}^{d+1}$ , classify all possible  $d \times d$  matrices  $A \in \mathbb{Z}^{d \times d}$  which are in Hermite normal form with  $\delta(\mathcal{P}) = (\delta_0, \delta_1, \dots, \delta_d) = v$ , where  $\mathcal{P} \subset \mathbb{R}^d$  is the integral simplex whose vertices are the row vectors of  $A$  together with the origin in  $\mathbb{R}^d$ .

**1.4. Structure of this paper.** In Section 2, the way we approach Problem 1.2 will be described. Concretely, we develop an algorithm for any Hermite normal form  $A$  to compute its  $\delta$ -vector. (See Theorem 2.1.) This actually gives a new way to compute the  $\delta$ -vector for any integral simplex via its Hermite normal form. This algorithm can be very efficient for simplices with small volumes and prime volumes.

Based on this algorithm, as a by-product, we can derive some conditions for Hermite normal forms to have “shifted symmetric”  $\delta$ -vector, namely,  $\delta_i = \delta_{d+1-i}$  for  $1 \leq i \leq d$ . We will discuss these conditions for two classes of Hermite normal forms in Section 3.

In Section 4, we apply Theorem 2.1 and obtain a solution to Problem 1.2 when  $\sum_{i=0}^d \delta_i \leq 4$ . Section 4.1 is devoted to studying the case  $\sum_{i=0}^d \delta_i = 2$ , Section 4.2 is  $\sum_{i=0}^d \delta_i = 3$  and Section 4.3 is  $\sum_{i=0}^d \delta_i = 4$ .

Finally, in section 5, as our main result, we show that the inequalities (2) and (3) with an additional condition will give all possible  $\delta$ -vectors with  $\sum_{i=0}^d \delta_i = 4$ . And in this case, all  $\delta$ -vectors can be obtained by simplices (Theorem 5.1).

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## 2. AN ALGORITHM FOR THE COMPUTATION OF THE $\delta$ -VECTOR OF A SIMPLEX

In this section, we introduce an algorithm for calculating the  $\delta$ -vector of integral simplices arising from Hermite normal forms.

Let  $M \in \mathbb{Z}^{d \times d}$ . We write  $\mathcal{P}(M)$  for the integral simplex whose vertices are the row vectors of  $M$  together with the origin in  $\mathbb{R}^d$ . We will present an algorithm to compute the  $\delta$ -vector of  $\mathcal{P}(M)$ . To make the notation clear, we assume  $d = 3$ . The general case is completely analogous. Let  $A$  be the Hermite normal form of  $M$ . We have that  $\{\mathcal{P}(M) \cap \mathbb{Z}^d\}$  is in bijection with  $\{\mathcal{P}(A) \cap \mathbb{Z}^d\}$ . By definition,

$$A = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

where each  $a_{ij}$  is a nonnegative integer.

For a vector  $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ , consider

$$b(\lambda) := (\lambda_1, \lambda_2, \lambda_3)A = (a_{11}\lambda_1 + a_{21}\lambda_2 + a_{31}\lambda_3, a_{22}\lambda_2 + a_{32}\lambda_3, a_{33}\lambda_3).$$

Then it is clear that the set of interior points inside  $\mathcal{P}(A)$  ( $(\mathcal{P}(A) - \partial\mathcal{P}(A)) \cap \mathbb{Z}^3$ ) is in bijection with the set

$$\{(\lambda_1, \lambda_2, \lambda_3) \mid \lambda_i > 0, \lambda_1 + \lambda_2 + \lambda_3 < 1, b(\lambda) \in \mathbb{Z}^3\}.$$

An observation is that  $n(\mathcal{P}(A) - \partial\mathcal{P}(A)) \cap \mathbb{Z}^3$ , for any  $n \in \mathbb{N}$ , is in bijection with

$$\{(\lambda_1, \lambda_2, \lambda_3) \mid \lambda_i > 0, \lambda_1 + \lambda_2 + \lambda_3 < n, b(\lambda) \in \mathbb{Z}^3\}.$$

We first consider all positive vectors  $\lambda$  satisfying  $b(\lambda) \in \mathbb{Z}^3$ . By the lower triangularity of the Hermite normal form, we can start from the last coefficient of  $b(\lambda)$  and move forward. It is not hard to see that each vector  $\lambda$  should have the following form: ( $\{r\}$  is the fractional part of a rational number  $r$ .)

$$\begin{aligned} \lambda_3 &= \lambda_3^{k,k_3} := \frac{k}{a_{33}} + k_3, \\ \lambda_2 &= \lambda_2^{j,k,k_2} := \frac{j - \{a_{32}\lambda_3^{k,k_3}\}}{a_{22}} + k_2, \end{aligned}$$

and

$$\lambda_1 = \lambda_1^{ijk, k_1} := \frac{i - \{a_{21}\lambda_2^{jk} + a_{31}\lambda_3^k\}}{a_{11}} + k_1,$$

for some nonnegative integers  $k_3, k_2, k_1$ , where  $k \in \{1, 2, \dots, a_{33}\}$ ,  $j \in \{1, 2, \dots, a_{22}\}$ ,  $i \in \{1, 2, \dots, a_{11}\}$  and  $\lambda_1^{ijk} = \lambda_1^{ijk, 0}$ ,  $\lambda_2^{jk} = \lambda_2^{jk, 0}$ ,  $\lambda_3^k = \lambda_3^{k, 0}$ . We call all the vectors  $\lambda$  with the same index  $(i, j, k)$  the *congruence class* of  $(i, j, k)$ .

Now we go to the condition  $\lambda_1 + \lambda_2 + \lambda_3 < n$  in the above bijection. As  $n$  increases, we ask when is the first time that a congruence class  $(i, j, k)$  starts to produce interior points inside  $n\mathcal{P}(A)$ . In other words, fix  $(i, j, k)$ . We want the smallest  $n$  such that  $\lambda_1 + \lambda_2 + \lambda_3 < n$  with  $\lambda_1, \lambda_2, \lambda_3 > 0$ . Then it is clear that this happens when  $k_1 = k_2 = k_3 = 0$  and

$$n = s_{ijk} := \lfloor \lambda_1^{ijk} + \lambda_2^{jk} + \lambda_3^k \rfloor + 1,$$

where  $\lfloor r \rfloor$  for a rational number is the biggest integer not larger than  $r$ .

Finally, when  $n$  grows larger than  $s_{ijk}$ , we want to consider how many interior points this fixed congruence class produces. Let  $n = s_{ijk} + \ell$ , so each interior point corresponds to a choice of  $k_1 \geq 0, k_2 \geq 0, k_3 \geq 0$  in the formula of  $\lambda_1^{ijk, k_1}$ ,  $\lambda_2^{ij, k_2}$  and  $\lambda_3^{i, k_3}$  such that  $k_1 + k_2 + k_3 \leq \ell$ . There are  $\binom{d+1}{\ell} = \binom{d+\ell}{\ell}$  choices in total.

To sum up, we have the following two observations for each congruence class  $(i, j, k)$ ,  $k \in \{1, 2, \dots, a_{33}\}$ ,  $j \in \{1, 2, \dots, a_{22}\}$ ,  $i \in \{1, 2, \dots, a_{11}\}$ :

- (1)  $s_{ijk}$  is the smallest  $n$  such that this congruence class contributes interior points in the  $n$ -th dilation of  $\mathcal{P}(A)$ ;
- (2) In the  $(s_{ijk} + \ell)$ -th dilation of  $\mathcal{P}(A)$ , this congruence class contributes  $\binom{d+1}{\ell}$  interior points.

Therefore, the following Theorem holds. We state it for a general dimension  $d$ , and the proof is analogous to the case  $d = 3$ .

**Theorem 2.1.** *Let  $\mathcal{P}(A)$  be a  $d$ -dimensional simplex corresponding to a  $d \times d$  matrix  $A = (a_{ij}) \in \mathbb{Z}^{d \times d}$ . Then the generating function for the interior points of  $n\mathcal{P}(A)$ ,  $i^*(\mathcal{P}(A), n) = |n(\mathcal{P}(A) - \partial\mathcal{P}(A)) \cap \mathbb{Z}^d|$  is*

$$\sum_{n=1}^{\infty} i^*(\mathcal{P}(A), n) t^n = (1-t)^{-(d+1)} \sum_{\substack{(i_1, \dots, i_d) \\ 1 \leq i_j \leq a_{ij}}} t^{s_{i_1 \dots i_d}},$$

where

$$s_{i_1 \dots i_d} = \left\lfloor \sum_{k=1}^d \lambda_k^{i_k, i_{k+1}, \dots, i_d} \right\rfloor + 1,$$

with

$$\lambda_d^{i_d} = \frac{i_d}{a_{dd}},$$

and for each  $1 \leq k < d$ ,

$$\lambda_k^{i_k, i_{k+1}, \dots, i_d} = a_{kk}^{-1} \left( i_k - \left\{ \sum_{h=k+1}^d a_{hk} \lambda_h^{i_h i_{h+1} \dots i_d} \right\} \right).$$

By the reciprocity law (0.3), we have

$$\delta_{\mathcal{P}(A)}(t) = \sum_{\substack{(i_1, \dots, i_d) \\ 1 \leq i_j \leq a_{ij}}} t^{d+1-s_{i_1 \dots i_d}}.$$

**Example 2.2.** Let  $A$  be the  $4 \times 4$  matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 0 & 1 & 3 \end{pmatrix}.$$

Consider

$$b(\lambda) = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)A = (\lambda_1 + \lambda_3 + \lambda_4, \lambda_2 + \lambda_3, 2\lambda_3 + \lambda_4, 3\lambda_4).$$

Denote

$$\begin{aligned} \lambda_4^j &= \frac{j}{3}, \text{ for } j = 1, 2, 3, & \lambda_3^{ij} &= \frac{i - \{\lambda_4^j\}}{2}, \text{ for } i = 1, 2, \\ \lambda_2^{ij} &= 1 - \{\lambda_3^{ij}\}, & \lambda_1^{ij} &= 1 - \{\lambda_3^{ij} + \lambda_4^j\} \end{aligned}$$

and

$$s_{ij} = 1 + \lfloor \lambda_1^{ij} + \lambda_2^{ij} + \lambda_3^{ij} + \lambda_4^j \rfloor.$$

Then we have

$$s_{11} = 2, s_{21} = 3, s_{12} = 2, s_{22} = 3, s_{13} = 3, s_{23} = 5,$$

$$\delta_{\mathcal{P}(A)}(t) = \sum_{i=1}^3 \sum_{j=1}^2 t^{d+1-s_{ij}} = 1 + 3t^2 + 2t^3,$$

and thus

$$\delta(\mathcal{P}(A)) = (1, 0, 3, 2, 0).$$

### 3. SHIFTED SYMMETRIC $\delta$ -VECTORS

In this section, we define shifted symmetric  $\delta$ -vectors and study its conditions for some special Hermite normal forms. Results in this section are direct applications of the algorithm developed in the previous section (Theorem 2.1). In [5], the second author studied shifted symmetric  $\delta$ -vectors without using the algorithm.

We call a  $\delta$ -vector *shifted symmetric*, if  $\delta_i = \delta_{d+1-i}$ ,  $1 \leq i \leq d$ . For example,  $(1, 1, 2, 2, 1, 2, 2, 1)$  is shifted symmetric.

We want this definition because it simply arises from the algorithm for the “one row” Hermite normal forms as discussed in the first subsection. In the second

subsection, we will consider a special “one row” Hermite normal form, which allows us to have better results.

**3.1. “One row” Hermite normal forms.** Consider all  $d \times d$  matrices with determinant  $D$  and the following Hermite normal forms for some  $k \in \{1, 2, \dots, d\}$ .

$$(4) \quad A_D = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ a_1 & \cdots & a_{k-1} & D & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix},$$

where  $a_1, \dots, a_{k-1}$  are nonnegative integers smaller than  $D$  and all other terms are zero. Let  $d_j$  denote the number of  $j$ 's among these  $a_\ell$ 's, for  $j = 1, \dots, D-1$ . Then we can simplify Theorem 2.1 for these “one row” Hermite normal forms.

**Corollary 3.1.** *Let  $M \in \mathbb{Z}^{d \times d}$  with  $\det(M) = D$  and  $\mathcal{P}(M)$  be the corresponding integral simplex. If its Hermite normal form is of the form as in (4), then we have*

$$\delta_{\mathcal{P}(M)}(t) = \sum_{i=1}^D t^{d+1-s_i},$$

where

$$(5) \quad s_i = \left\lfloor \frac{i}{D} - \sum_{j=1}^{D-1} \left\{ \frac{ij}{D} \right\} d_j \right\rfloor + d.$$

*Proof.* Consider

$$b(\lambda) = (\lambda_1, \dots, \lambda_k, \dots, \lambda_d) A_D = (\lambda_1 + a_1 \lambda_k, \dots, \lambda_{k-1} + a_{k-1} \lambda_k, D \lambda_k, \lambda_{k+1}, \dots, \lambda_d).$$

Using notation from the proof of Theorem 2.1, we have, for  $i = 1, 2, \dots, D$ ,

$$\lambda_k^i = \frac{i}{D}, \quad \lambda_\ell^i = 1 - \left\{ a_\ell \frac{i}{D} \right\}, \quad \text{for } \ell = 1, \dots, k-1$$

and

$$\lambda_{k+1}^i = \dots = \lambda_d^i = 1.$$

$$\text{Therefore, } s_i = 1 + \lfloor \lambda_1^i + \dots + \lambda_d^i \rfloor = \left\lfloor \frac{i}{D} - \sum_{j=1}^{D-1} \left\{ \frac{ij}{D} \right\} d_j \right\rfloor + d. \quad \square$$

Now we are going to deduce a symmetry property of the  $\delta$ -vectors by using this Corollary.

**Proposition 3.2** (Shifted symmetry for “one row”). *For a matrix  $M \in \mathbb{Z}^{d \times d}$  with Hermite normal form (4), we have  $s_i + s_{D-i} = d + 1$ , for  $i = 1, \dots, D - 1$ , which implies  $\delta_i = \delta_{d+1-i}$  by reciprocity, if and only if the following three conditions hold:*

- (1)  $\sum_{j=1}^{D-1} jd_j - 1$  is coprime with  $D$ ;
- (2)  $d_j = 0$  for all  $j$  which is not coprime with  $D$ ;
- (3)  $\sum_{j=1}^{D-1} d_j = d - 1$ .

*Proof.* Let us consider  $s_i + s_{D-i}$ . For an integer  $a$ , let  $\bar{a}$  denote the residue class in  $\mathbb{Z}/D\mathbb{Z}$ . Then we have

$$\begin{aligned} s_i + s_{D-i} &= \left\lfloor \frac{i}{D} - \sum_{j=1}^{D-1} \left\{ \frac{ij}{D} \right\} d_j \right\rfloor + \left\lfloor \frac{D-i}{D} - \sum_{j=1}^{D-1} \left\{ \frac{(D-i)j}{D} \right\} d_j \right\rfloor + 2d \\ &= \left\lfloor \frac{i - \sum_{j=1}^{D-1} \overline{ij} d_j}{D} \right\rfloor + \left\lfloor \frac{D-i - \sum_{j=1}^{D-1} \overline{(D-i)j} d_j}{D} \right\rfloor + 2d. \end{aligned}$$

Since

$$(6) \quad \begin{cases} i - \sum_{j=1}^{D-1} \overline{ij} d_j \equiv i \left( 1 - \sum_{j=1}^{D-1} jd_j \right) \pmod{D}, \\ D-i - \sum_{j=1}^{D-1} \overline{(D-i)j} d_j \equiv (D-i) \left( 1 - \sum_{j=1}^{D-1} jd_j \right) \pmod{D}, \end{cases}$$

if the condition (1) is not satisfied, then one has

$$\begin{aligned} s_i + s_{D-i} &= \frac{D - \sum_{j=1}^{D-1} \left( \overline{ij} + \overline{(D-i)j} \right) d_j}{D} + 2d \\ &= 2d + 1 - \sum_{j=1}^{D-1} \frac{\overline{ij} + \overline{(D-i)j}}{D} d_j \\ &\geq 2d + 1 - \sum_{j=1}^{D-1} d_j \geq d + 2 > d + 1 \end{aligned}$$

for some  $i$  with  $1 \leq i \leq D - 1$ . Thus, the condition (1) is a necessary condition to be  $s_i + s_{D-i} = d + 1$  for all  $i$ . On the contrary, when the condition (1) is satisfied, again from (6), we have

$$\begin{aligned} s_i + s_{D-i} &= \frac{D - \sum_{j=1}^{D-1} \left( \overline{ij} + \overline{(D-i)j} \right) d_j}{D} + 2d - 1 \\ &= 2d - \sum_{j=1}^{D-1} \frac{\overline{ij} + \overline{(D-i)j}}{D} d_j \\ &= 2d - \sum_{D \text{ does not divide } ij} d_j. \end{aligned}$$



If the condition (2) is not satisfied, then we have

$$s_i + s_{D-i} = 2d - \sum_{D \text{ does not divide } ij} d_j > d + 1$$

for some  $i$  with  $1 \leq i \leq D-1$ . Hence, the condition (2) is also a necessary condition. In addition, if the condition (3) is not satisfied, then we have  $s_i + s_{D-i} > d + 1$ . Thus, the condition (3) is also a necessary condition. On the other hand, when the conditions (1), (2) and (3) are all satisfied, we have  $s_i + s_{D-i} = D + 1$  for all  $i$ .

Therefore, we obtain a necessary and sufficient to be  $s_i + s_{D-i} = d + 1$  for all  $i$ .

□

The conditions of Proposition 3.2 are not very easy to check, so we consider a special case of Hermite normal forms (4).

**3.2. “All  $D - 1$  one row” Hermite normal forms.** Assume in addition  $d_{D-1} = d - 1$  in Corollary 3.1, i.e., the Hermite normal form looks like

$$(7) \quad \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ D-1 & D-1 & \cdots & D-1 & D \end{pmatrix}.$$

Then we have

**Corollary 3.3** (All  $D - 1$ ). *For a matrix  $M \in \mathbb{Z}^{d \times d}$  with Hermite normal form (7), we have*

$$\delta_{\mathcal{P}(M)}(t) = \sum_{i=1}^D t^{d+1-s_i}, \text{ where } s_i = \left\lfloor \frac{id}{D} \right\rfloor + 1.$$

For the Hermite normal form (7), the conditions for shifted symmetry in Proposition 3.2 can be simplified.

**Proposition 3.4** (Shifted symmetry for “all  $D - 1$  one row”). *Let  $M \in \mathbb{Z}^{d \times d}$  with Hermite normal form (7). Then*

- (1)  $\delta_i = \delta_{d+1-i}$  if and only if  $D$  and  $d$  are coprime.
- (2) When  $D = kd$ , for  $k \in \mathbb{N}$  and  $k \geq 2$ , the  $\delta$ -vector is

$$(1, \underbrace{k, \dots, k}_{d-1}, k-1),$$

which is not shifted symmetric. But for  $k = 2$ , we have  $\delta_k = \delta_{d-k}$  (Gorenstein).

#### 4. CLASSIFICATION OF HERMITE NORMAL FORMS WITH A GIVEN $\delta$ -VECTOR

In this section, we will give another application of the algorithm Theorem 2.1. Consider Problem 1.2 first with the assumption that matrix  $A \in \mathbb{Z}^{d \times d}$  has prime determinant, i.e.,  $A$  is of the form (4), with only one general row. By Corollary 3.1, in order to classify all possible Hermite normal forms (4) with a given  $\delta$ -vector  $(\delta_0, \delta_1, \dots, \delta_d)$ , we need to find all nonnegative integer solutions  $(d_1, d_2, \dots, d_{D-1})$  with  $d_1 + d_2 + \dots + d_{D-1} \leq d - 1$  such that

$$\#\{i : d + 1 - s_i = j, \text{ for } i = 1, \dots, D\} = \delta_j, \text{ for } j = 0, \dots, d.$$

By Corollary 3.1, we can build equations with “floor” expressions for  $(d_1, d_2, \dots, d_{D-1})$ . Remove the “floor” expressions, we obtain  $D$  linear equations of  $(d_1, d_2, \dots, d_{D-1})$  with different constant terms but the same  $D \times D$  coefficient matrix  $M$  with  $ij$  entry  $\{(ij) \bmod D\}$ , which is a number in  $\{0, 1, \dots, D - 1\}$ . Then we first get all integer solutions  $(d_1, d_2, \dots, d_{D-1})$ , and then test every candidates by the restrictions of nonnegativity and  $d_1 + d_2 + \dots + d_{D-1} \leq d - 1$ .

For  $D = 2$  and  $3$ , the coefficient matrix  $M$  is nonsingular, so we can write down the complete solutions, as presented in the first two subsections. For larger primes, the coefficient matrix becomes singular, so there are free variables in the integer solutions  $(d_1, d_2, \dots, d_{D-1})$ , which make it very hard to simplify the final solutions after the test.

The idea is similar for Hermite normal forms with non prime determinant. Instead of using Corollary 3.1, we need to use the formulas in Theorem 2.1. In the third subsection, we will present the complete solution for  $D = 4$ .

**4.1. A solution of Problem 1.2 when  $\sum_{i=0}^d \delta_i = 2$ .** The goal of this subsection is to give a solution of Problem 1.2 when  $\sum_{i=0}^d \delta_i = 2$ , i.e., given a  $\delta$ -vector  $\delta(\mathcal{P})$  with  $\sum_{i=0}^d \delta_i = 2$ , we classify all the integral simplices with  $\delta(\mathcal{P})$  arising from Hermite normal forms with determinant 2.

We consider all Hermite normal forms (4) with  $D = 2$ , namely,

$$(8) \quad A_2 = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ * & \cdots & * & 2 & \\ & & & 1 & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix},$$

where there are  $d_1$  1's among the  $*$ 's. Notice that the position of the row with a 2 does not affect the  $\delta$ -vector, so the only variable is  $d_1$ . By Corollary 3.1, we have a

formula for the  $\delta$ -vector of this integral simplex  $\mathcal{P}(A_2)$ . Denote

$$k = 1 - \left\lfloor \frac{1 - d_1}{2} \right\rfloor.$$

Then one has  $\delta_0 = \delta_k = 1$ .

By this formula, we can characterize all Hermite normal forms with a given  $\delta$ -vector. Let  $\delta_0 = \delta_i = 1$ . Then by solving the equation  $i = 1 - \left\lfloor \frac{1 - d_1}{2} \right\rfloor$ , we obtain  $d_1 = 2i - 2$  and  $d_1 = 2i - 1$ , both cases will give us the desired  $\delta$ -vector.

Notice that there is a constraint on  $d_1$  to be  $0 \leq d_1 \leq d - 1$ . Not all  $\delta$ -vectors are obtained by from simplices. But we can easily get the restriction of  $i$  and the corresponding  $d_1$  as follows (by  $d_1 \geq 0$ , we have  $i \geq 1$ ):

- (1) If  $i \leq d/2$ ,  $d_1 = 2i - 2$  and  $d_1 = 2i - 1$  both work, and these give all the matrices with this  $\delta$ -vector.
- (2) If  $i = (d + 1)/2$ , only  $d_1 = 2i - 2 = d - 1$  works.
- (3) If  $i > (d + 1)/2$ , there is no solution.

Now, this result has been obtained essentially in [4]. In fact, the inequality  $i \leq (d + 1)/2$  means that the  $\delta$ -vector satisfies (3).

**4.2. A solution of Problem 1.2 when  $\sum_{i=0}^d \delta_i = 3$ .** We consider all Hermite normal forms (4) with  $D = 3$ , namely,

$$(9) \quad A_3 = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ * & \cdots & * & 3 & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix},$$

where there are  $d_1$  1's and  $d_2$  2's among the  $*$ 's. Since the position of the row with a 3 does not affect the  $\delta$ -vector, so the only variables are  $d_1$  and  $d_2$ . Also, by Corollary 3.1, we have  $\delta_{\mathcal{P}(A_3)}(t) = 1 + t^{k_1} + t^{k_2}$ , where

$$k_1 = 1 - \left\lfloor \frac{1 - d_1 - 2d_2}{3} \right\rfloor \text{ and } k_2 = 1 - \left\lfloor \frac{2 - 2d_1 - d_2}{3} \right\rfloor.$$

Then by the formula, similar to the case of  $\sum_{i=0}^d \delta_i = 2$ , though a little more complicated, we can characterize all Hermite normal forms with a given  $\delta$ -vector. Let  $\delta_{\mathcal{P}(A_3)}(t) = 1 + t^i + t^j$ . Set

$$i = 1 - \left\lfloor \frac{1 - d_1 - 2d_2}{3} \right\rfloor \text{ and } j = 1 - \left\lfloor \frac{2 - 2d_1 - d_2}{3} \right\rfloor.$$

(Later reverse the role of  $i$  and  $j$  if  $i \neq j$ , in both equations and solutions.) After computations, the solutions for  $(d_1, d_2)$  are

$$d^{(1)} = \begin{cases} d_1 = 2j - i \\ d_2 = 2i - j - 1, \end{cases} \quad d^{(2)} = \begin{cases} d_1 = 2j - i - 1 \\ d_2 = 2i - j - 1 \end{cases} \quad \text{and} \quad d^{(3)} = \begin{cases} d_1 = 2j - i \\ d_2 = 2i - j - 2. \end{cases}$$

In addition, by the restriction on  $(d_1, d_2)$  that  $d_1, d_2 \geq 0$  and  $d_1 + d_2 \leq d - 1$ , we have the following characterizations:

TABLE 1. Characterizations for matrices of the form (9)

$2j$	$2i$	$i + j$	solutions
$\geq i$	$\geq j + 1$	$\leq d$	$d^{(1)}$
$\geq i + 1$	$\geq j + 1$	$\leq d + 1$	$d^{(2)}$
$\geq i$	$\geq j + 2$	$\leq d + 1$	$d^{(3)}$

- (1) If  $2j \geq i, 2i \geq j + 1$  and  $i + j \leq d$ , then the solution  $d^{(1)}$  will work and this gives all the matrices with this  $\delta$ -vector.
- (2) If  $2j \geq i + 1, 2i \geq j + 1$  and  $i + j \leq d + 1$ , then the solution  $d^{(2)}$  will work and this gives all the matrices with this  $\delta$ -vector.
- (3) If  $2j \geq i, 2i \geq j + 2$  and  $i + j \leq d + 1$ , then the solution  $d^{(3)}$  will work and this gives all the matrices with this  $\delta$ -vector.
- (4) If  $\{i, j\}$  in the given vector does not satisfy any of the above cases, there is no matrix with this vector as its  $\delta$ -vector.

Again, this result has been obtained in [4]. In fact, for example, the inequality  $2j \geq i$  means that (2) holds and the inequality  $i + j \leq d + 1$  means that (3) holds.

Notice that only the solution

$$d^{(2)} = \begin{cases} d_1 = d - 1 \\ d_2 = 0 \end{cases}$$

works when  $i = (d + 2)/3$  and  $j = (2d + 1)/3$ . This happens when  $d \equiv 1 \pmod{3}$  and there is only one matrix with  $d_1 = d - 1$  and  $d_2 = 0$ . Similary, only the solution

$$d^{(3)} = \begin{cases} d_1 = 0 \\ d_2 = d - 1 \end{cases}$$

works when  $i = (2d + 2)/3$  and  $j = (d + 1)/3$ . This happens when  $d \equiv 2 \pmod{3}$  and again, there is only one matrix with  $d_1 = 0$  and  $d_2 = d - 1$ .

**4.3. A solution of Problem 1.2 when  $\sum_{i=0}^d \delta_i = 4$ .** When the determinant is 4, there are two cases of Hermite normal forms. One is the Hermite normal forms (4)

with  $D = 4$ , namely,

$$(10) \quad A_4 = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ * & \cdots & * & 4 & & \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix},$$

where there are  $d_1$  1's,  $d_2$  2's and  $d_3$  3's among  $*$ 's. The other one looks like

$$(11) \quad A'_4 = \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ * & \cdots & * & 2 & & & & \\ & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & & & & 1 & \\ * & \cdots & * & \bar{*} & * & \cdots & * & 2 \\ & & & & & & & 1 \\ & & & & & & & & \ddots \end{pmatrix},$$

where there are  $d_1$  1's (resp.  $d'_1$  1's) among  $*$ 's (resp.  $\bar{*}$ 's), there are  $e_1$  1's (resp.  $e'_1$  1's) among  $\bar{*}$ 's (resp.  $*$ 's) of which the entry of the row of  $\bar{*}$  (resp.  $*$ ) in the same column is 0. Also, set  $d''_1 = e_1 + e'_1$ . (For example, a  $6 \times 6$  Hermite normal form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 2 \end{pmatrix}$$

is a matrix (11) with  $d_1 = 2, d'_1 = 3, e_1 = 1, e'_1 = 2, d''_1 = 3$  and  $\bar{*} = 1$ .)

First, we consider the Hermite normal forms (10). Then, by Corollary 3.1, we have  $\delta_{\mathcal{P}(A_4)}(t) = 1 + t^{k_1} + t^{k_2} + t^{k_3}$ , where

$$k_1 = 1 - \left\lfloor \frac{1 - d_1 - 2d_2 - 3d_3}{4} \right\rfloor, k_2 = 1 - \left\lfloor \frac{1 - d_1 - d_3}{2} \right\rfloor \text{ and } k_3 = 1 - \left\lfloor \frac{3 - 3d_1 - 2d_2 - d_3}{4} \right\rfloor.$$

Let  $\delta_{\mathcal{P}(A_4)}(t) = 1 + t^i + t^j + t^k$ . We get three sets of equations, according to the order of  $k_1, k_2$  and  $k_3$ :

$$i = 1 - \left\lfloor \frac{1 - d_1 - 2d_2 - 3d_3}{4} \right\rfloor, j = 1 - \left\lfloor \frac{1 - d_1 - d_3}{2} \right\rfloor \text{ and } k = 1 - \left\lfloor \frac{3 - 3d_1 - 2d_2 - d_3}{4} \right\rfloor.$$

(Later replace the roles of  $i, j$  and  $k$  if any of the three are distinct.) After computations, the solutions for  $(d_1, d_2, d_3)$  are

$$d^{(1)} = \begin{cases} d_1 = -i + j + k - 1 \\ d_2 = i - 2j + k \\ d_3 = i + j - k - 1, \end{cases} \quad d^{(2)} = \begin{cases} d_1 = -i + j + k \\ d_2 = i - 2j + k \\ d_3 = i + j - k - 2, \end{cases}$$

$$d^{(3)} = \begin{cases} d_1 = -i + j + k \\ d_2 = i - 2j + k \\ d_3 = i + j - k - 1 \end{cases} \quad \text{and} \quad d^{(4)} = \begin{cases} d_1 = -i + j + k \\ d_2 = i - 2j + k - 1 \\ d_3 = i + j - k - 1. \end{cases}$$

In addition, by the restriction on  $(d_1, d_2, d_3)$  that  $d_1, d_2, d_3 \geq 0$  and  $d_1 + d_2 + d_3 \leq d - 1$ , we have the following characterizations:

TABLE 2. Characterizations for matrices of the form (10)

$j + k$	$2j$	$i + j$	solutions
$\geq i + 1$	$\leq i + k \leq d + 1$	$\geq k + 1$	$d^{(1)}$
$\geq i$	$\leq i + k \leq d + 1$	$\geq k + 2$	$d^{(2)}$
$\geq i$	$\leq i + k \leq d$	$\geq k + 1$	$d^{(3)}$
$\geq i$	$\leq i + k - 1 \leq d$	$\geq k + 1$	$d^{(4)}$

- (1) If  $j + k \geq i + 1, 2j \leq i + k \leq d + 1$  and  $i + j \geq k + 1$ , then the solution  $d^{(1)}$  will work and this gives all the matrices with this  $\delta$ -vector.
- (2) If  $j + k \geq i, 2j \leq i + k \leq d + 1$  and  $i + j \geq k + 2$ , then the solution  $d^{(2)}$  will work and this gives all the matrices with this  $\delta$ -vector.
- (3) If  $j + k \geq i, 2j \leq i + k \leq d$  and  $i + j \geq k + 1$ , then the solution  $d^{(3)}$  will work and this gives all the matrices with this  $\delta$ -vector.
- (4) If  $j + k \geq i, 2j + 1 \leq i + k \leq d + 1$  and  $i + j \geq k + 1$ , then the solution  $d^{(4)}$  will work and this gives all the matrices with this  $\delta$ -vector.
- (5) If  $\{i, j, k\}$  in the given vector does not satisfy any of the above cases, there is no matrix (10) with this vector as its  $\delta$ -vector.

Notice that only the solution

$$d^{(2)} = \begin{cases} d_1 = 0 \\ d_2 = 0 \\ d_3 = d - 1 \end{cases}$$

works when  $i = (3d + 3)/4, j = (d + 1)/2$  and  $k = (d + 1)/4$ . This happens when  $d \equiv 3 \pmod{4}$  and there is only one matrix with  $d_3 = d - 1$ . Similarly, only the

solution

$$d^{(1)} = \begin{cases} d_1 = d - 1 \\ d_2 = 0 \\ d_3 = 0 \end{cases}$$

works when  $i = (d + 3)/4, j = (d + 1)/2$  and  $k = (3d + 1)/4$ . This happens when  $d \equiv 1 \pmod{4}$  and again, there is only one matrix with  $d_1 = d - 1$ .

Next, we consider the Hermite normal forms (11). However, we need to consider two cases, which are the cases where  $\bar{*} = 0$  and  $\bar{*} = 1$ .

First, we consider the case with  $\bar{*} = 0$ . Notice that the variables are  $d_1, d'_1$  and  $d''_1$ . Obviously we cannot use Corollary 3.1, but we apply Theorem 2.1 directly. Thus we have  $\delta_{\mathcal{P}(A'_4)}(t) = 1 + t^{k_1} + t^{k_2} + t^{k_3}$ , where

$$k_1 = \left\lfloor \frac{d_1 + 2}{2} \right\rfloor, \quad k_2 = \left\lfloor \frac{d'_1 + 2}{2} \right\rfloor \text{ and } k_3 = \left\lfloor \frac{d''_1 + 3}{2} \right\rfloor.$$

Let  $\delta_{\mathcal{P}(A'_4)}(t) = 1 + t^i + t^j + t^k$ . We get three sets of equations, according to the order of  $k_1, k_2$  and  $k_3$ :

$$i = \left\lfloor \frac{d_1 + 2}{2} \right\rfloor, \quad j = \left\lfloor \frac{d'_1 + 2}{2} \right\rfloor \text{ and } k = \left\lfloor \frac{d''_1 + 3}{2} \right\rfloor.$$

or replace the role of  $i, j$  and  $k$  if  $i, j$  and  $k$  are distinct, in all equations and solutions. After computations, since  $d_1 + d'_1 + d''_1$  is even, the solutions of  $(d_1, d'_1, d''_1)$  are

$$\begin{aligned} d^{(1)} &= \begin{cases} d_1 = 2i - 2 \\ d'_1 = 2j - 1 \\ d''_1 = 2k - 3, \end{cases} & d^{(2)} &= \begin{cases} d_1 = 2i - 1 \\ d'_1 = 2j - 2 \\ d''_1 = 2k - 3, \end{cases} \\ d^{(3)} &= \begin{cases} d_1 = 2i - 1 \\ d'_1 = 2j - 1 \\ d''_1 = 2k - 2 \end{cases} & \text{and } d^{(4)} &= \begin{cases} d_1 = 2i - 2 \\ d'_1 = 2j - 2 \\ d''_1 = 2k - 2. \end{cases} \end{aligned}$$

In addition, by the restriction on  $(d_1, d'_1, d''_1)$  that  $0 \leq d_1 \leq d - 2, 0 \leq d'_1 \leq d - 2, 0 \leq d''_1 \leq d - 2, d_1 + d'_1 + d''_1 \leq 2(d - 2), d''_1 \leq d_1 + d'_1, d'_1 \leq d_1 + d''_1$  and  $d_1 \leq d'_1 + d''_1$ , we have the following characterizations:

- (1) If  $i \leq \lfloor d/2 \rfloor, j \leq \lfloor (d - 1)/2 \rfloor, 2 \leq k \leq \lfloor (d + 1)/2 \rfloor, i + j + k \leq d + 1, k \leq i + j, j + 2 \leq i + k$  and  $i + 1 \leq j + k$ , then the solution  $d^{(1)}$  will work and this gives all the matrices with this  $\delta$ -vector.
- (2) If  $i \leq \lfloor (d - 1)/2 \rfloor, j \leq \lfloor d/2 \rfloor, 2 \leq k \leq \lfloor (d + 1)/2 \rfloor, i + j + k \leq d + 1, k \leq i + j, j + 1 \leq i + k$  and  $i + 2 \leq j + k$ , then the solution  $d^{(2)}$  will work and this gives all the matrices with this  $\delta$ -vector.

TABLE 3. Characterizations for matrices of the form (11) with  $\bar{*} = 0$

$i$	$j$	$k$	$i + j$	$i + k$	$j + k$	$i + j + k$	solutions
$\leq \left\lfloor \frac{d}{2} \right\rfloor$	$\leq \left\lfloor \frac{d-1}{2} \right\rfloor$	$\geq 2,$ $\leq \left\lfloor \frac{d+1}{2} \right\rfloor$	$\geq k$	$\geq j + 2$	$\geq i + 1$	$\leq d + 1$	$d^{(1)}$
$\leq \left\lfloor \frac{d-1}{2} \right\rfloor$	$\leq \left\lfloor \frac{d}{2} \right\rfloor$	$\geq 2,$ $\leq \left\lfloor \frac{d+1}{2} \right\rfloor$	$\geq k$	$\geq j + 1$	$\geq i + 2$	$\leq d + 1$	$d^{(2)}$
$\leq \left\lfloor \frac{d-1}{2} \right\rfloor$	$\leq \left\lfloor \frac{d-1}{2} \right\rfloor$	$\leq \left\lfloor \frac{d}{2} \right\rfloor$	$\geq k$	$\geq j + 1$	$\geq i + 1$	$\leq d$	$d^{(3)}$
$\leq \left\lfloor \frac{d}{2} \right\rfloor$	$\leq \left\lfloor \frac{d}{2} \right\rfloor$	$\leq \left\lfloor \frac{d}{2} \right\rfloor$	$\geq k + 1$	$\geq j + 1$	$\geq i + 1$	$\leq d + 1$	$d^{(4)}$

- (3) If  $k \leq \lfloor d/2 \rfloor$ ,  $i, j \leq \lfloor (d-1)/2 \rfloor$ ,  $i + j + k \leq d$ ,  $k \leq i + j$ ,  $j + 1 \leq i + k$  and  $i + 1 \leq j + k$ , then the solution  $d^{(3)}$  will work and this gives all the matrices with this  $\delta$ -vector.
- (4) If  $i, j, k \leq \lfloor d/2 \rfloor$ ,  $i + j + k \leq d + 1$ ,  $k + 1 \leq i + j$ ,  $j + 1 \leq i + k$  and  $i + 1 \leq j + k$ , then the solution  $d^{(4)}$  will work and this gives all the matrices with this  $\delta$ -vector.
- (5) If  $\{i, j, k\}$  in the given vector does not satisfy any of the above cases, there is no matrix (11), where  $\bar{*} = 0$ , with this vector as its  $\delta$ -vector.

Next, we consider the case with  $\bar{*} = 1$ . By Theorem 2.1, we have  $\delta_{\mathcal{P}(A'_4)}(t) = 1 + t^{k_1} + t^{k_2} + t^{k_3}$ , where

$$k_1 = 1 - \left\lfloor \frac{1 - d_1 - 2d_1''}{4} \right\rfloor, \quad k_2 = 1 - \left\lfloor \frac{1 - d_1}{2} \right\rfloor \quad \text{and} \quad k_3 = 2 - \left\lfloor \frac{3 - d_1 - 2d_1'}{4} \right\rfloor.$$

Let  $\delta_{\mathcal{P}(A'_4)}(t) = 1 + t^i + t^j + t^k$ . We get three sets of equations, according to the order of  $k_1, k_2$  and  $k_3$ :

$$i = 1 - \left\lfloor \frac{1 - d_1 - 2d_1''}{4} \right\rfloor, \quad j = 1 - \left\lfloor \frac{1 - d_1}{2} \right\rfloor \quad \text{and} \quad k = 2 - \left\lfloor \frac{3 - d_1 - 2d_1'}{4} \right\rfloor.$$



or replace the roles of  $i, j$  and  $k$  if  $i, j$  and  $k$  are distinct. After computations, considering  $d_1 + d'_1 + d''_1$  is even, the solutions of  $(d_1, d'_1, d''_1)$  are

$$d^{(1)} = \begin{cases} d_1 = 2j - 1 \\ d'_1 = 2k - j - 3 \\ d''_1 = 2i - j - 2, \end{cases} \quad d^{(2)} = \begin{cases} d_1 = 2j - 1 \\ d'_1 = 2k - j - 2 \\ d''_1 = 2i - j - 1, \end{cases}$$

$$d^{(3)} = \begin{cases} d_1 = 2j - 2 \\ d'_1 = 2k - j - 3 \\ d''_1 = 2i - j - 1 \end{cases} \quad \text{and} \quad d^{(4)} = \begin{cases} d_1 = 2j - 2 \\ d'_1 = 2k - j - 2 \\ d''_1 = 2i - j - 2. \end{cases}$$

In addition, by the restriction on  $(d_1, d'_1, d''_1)$  that  $0 \leq d_1 \leq d - 2$ ,  $0 \leq d'_1 \leq d - 2$ ,  $0 \leq d''_1 \leq d - 2$ ,  $d_1 + d'_1 + d''_1 \leq 2(d - 2)$ ,  $d''_1 \leq d_1 + d'_1$ ,  $d'_1 \leq d_1 + d''_1$  and  $d_1 \leq d'_1 + d''_1$ , we have the following characterizations:

TABLE 4. Characterizations for matrices of the form (11) with  $\bar{*} = 1$

$2k$	$2i$	$2j$	$i + j$	$i + k$	$j + k$	solutions
$\geq j + 3,$ $\leq d + j + 1$	$\geq j + 2,$ $\leq d + j$	$\leq d - 1$	$\geq k$	$\geq 2j + 2,$ $\leq d + 1$	$\geq i + 1$	$d^{(1)}$
$\geq j + 2,$ $\leq d + j$	$\geq j + 1,$ $\leq d + j - 1$	$\leq d - 1$	$\geq k$	$\geq 2j + 1,$ $\leq d$	$\geq i + 1$	$d^{(2)}$
$\geq j + 3,$ $\leq d + j + 1$	$\geq j + 1,$ $\leq d + j - 1$	$\leq d$	$\geq k$	$\geq 2j + 1,$ $\leq d + 1$	$\geq i + 2$	$d^{(3)}$
$\geq j + 2,$ $\leq d + j$	$\geq j + 2,$ $\leq d + j$	$\leq d$	$\geq k + 1$	$\geq 2j + 1,$ $\leq d + 1$	$\geq i + 1$	$d^{(4)}$

- (1) If  $j + 3 \leq 2k \leq d + j + 1$ ,  $j + 2 \leq 2i \leq d + j$ ,  $2j \leq d - 1$ ,  $2j + 2 \leq i + k \leq d + 1$ ,  $i + 1 \leq j + k$  and  $k \leq i + j$ , then the solution  $d^{(1)}$  will work and this gives all the matrices with this  $\delta$ -vector.
- (2) If  $j + 2 \leq 2k \leq d + j$ ,  $j + 1 \leq 2i \leq d + j - 1$ ,  $2j \leq d - 1$ ,  $2j + 1 \leq i + k \leq d$ ,  $i + 1 \leq j + k$  and  $k \leq i + j$ , then the solution  $d^{(2)}$  will work and this gives all the matrices with this  $\delta$ -vector.
- (3) If  $j + 3 \leq 2k \leq d + j + 1$ ,  $j + 1 \leq 2i \leq d + j - 1$ ,  $2j \leq d$ ,  $2j + 1 \leq i + k \leq d + 1$ ,  $i + 2 \leq j + k$  and  $k \leq i + j$  then the solution  $d^{(3)}$  will work and this gives all the matrices with this  $\delta$ -vector.
- (4) If  $j + 2 \leq 2k \leq d + j$ ,  $j + 2 \leq 2i \leq d + j$ ,  $2j \leq d$ ,  $2j + 1 \leq i + k \leq d + 1$ ,  $i + 1 \leq j + k$  and  $k + 1 \leq i + j$  then the solution  $d^{(4)}$  will work and this gives all the matrices with this  $\delta$ -vector.
- (5) If  $\{i, j, k\}$  in the given vector does not satisfy any of the above cases, there is no matrix (11) with this vector as its  $\delta$ -vector.

Notice that only the solution

$$d^{(3)} = \begin{cases} d_1 = d - 2 \\ d'_1 = d - 2 \\ d''_1 = 0 \end{cases}$$

works when  $i = (d + 2)/4, j = d/2$  and  $k = (3d + 2)/4$ . This happens when  $d \equiv 2 \pmod{4}$  and there is only one matrix with  $d_1 = d'_1 = d - 2$ . Similarly, only the solution

$$d^{(4)} = \begin{cases} d_1 = d - 2 \\ d'_1 = 0 \\ d''_1 = d - 2 \end{cases}$$

works when  $i = 3d/4, j = d/2$  and  $k = d/4 + 1$ . This happens when  $d \equiv 0 \pmod{4}$  and again, there is only one matrix with  $d_1 = d''_1 = d - 2$ .

## 5. THE CLASSIFICATION OF THE POSSIBLE $\delta$ -VECTORS WITH $\sum_{i=0}^d \delta_i = 4$

In this section, in consequence, we classify the possible  $\delta$ -vectors with  $\sum_{i=0}^d \delta_i = 4$  by results from Section 4.3.

Let  $1 + t^{i_1} + t^{i_2} + t^{i_3}$  with  $1 \leq i_1 \leq i_2 \leq i_3 \leq d$  be a  $\delta$ -polynomial for some integral polytope and  $(\delta_0, \delta_1, \dots, \delta_d)$  a sequence of the coefficients of this polynomial, where it is clear that  $\delta_0 = 1$  and  $\sum_{i=0}^d \delta_i = 4$ . Assume that  $(\delta_0, \delta_1, \dots, \delta_d)$  satisfies the inequalities (2), (3) and  $\delta_1 \geq \delta_d$ , which are necessary conditions to be a possible  $\delta$ -vector. Then (2) and (3) lead into the following inequalities that  $(i_1, i_2, i_3)$  satisfies

$$(12) \quad i_3 \leq i_1 + i_2, \quad i_1 + i_3 \leq d + 1 \quad \text{and} \quad i_2 \leq \lfloor (d + 1)/2 \rfloor.$$

Finally, the classification of possible  $\delta$ -vectors of integral polytopes with  $\sum_{i=0}^d \delta_i = 4$  is given by the following

**Theorem 5.1.** *Let  $1 + t^{i_1} + t^{i_2} + t^{i_3}$  be a polynomial with  $1 \leq i_1 \leq i_2 \leq i_3 \leq d$ . Then there exists an integral polytope  $\mathcal{P} \subset \mathbb{R}^d$  of dimension  $d$  whose  $\delta$ -polynomial equals  $1 + t^{i_1} + t^{i_2} + t^{i_3}$  if and only if  $(i_1, i_2, i_3)$  satisfies (12) and an additional condition*

$$(13) \quad 2i_2 \leq i_1 + i_3 \quad \text{or} \quad i_2 + i_3 \leq d + 1.$$

*Moreover, all these polytopes can be chosen to be simplices.*

*Proof.* There are four cases: (1)  $i_1 = i_2 = i_3$ , (2)  $i_1 < i_2 = i_3$ , (3)  $i_1 = i_2 < i_3$ , (4)  $i_1 < i_2 < i_3$ . We will show that in each case (12) together with (13) are the necessary and sufficient conditions for  $1 + t^{i_1} + t^{i_2} + t^{i_3}$  to be the  $\delta$ -vector of some integral polytope.

(1) Assume  $i_1 = i_2 = i_3 = \ell$ . By the inequalities (12), we have  $1 \leq \ell \leq \lfloor (d+1)/2 \rfloor$ . Set  $i = j = k = \ell$ . We have

$$(14) \quad j + k \geq i + 1, \quad 2j \leq i + k \leq d + 1 \text{ and } i + j \geq k + 1.$$

Thus, by our result on the classification of the case of a matrix (10) (Table 2, the solution  $d^{(1)}$ ), there exists an integral simplex whose  $\delta$ -vector is  $(1, 0, \dots, 0, 3, 0, \dots, 0)$ .

On the other hand, if there exists an integral polytope with this  $\delta$ -vector, then (12) holds since it is a necessary condition. In this case, it follows that both inequalities in (13) hold.

(2) Assume  $\ell = i_1 < i_2 = i_3 = \ell'$ . By (12), we have  $1 \leq \ell < \ell' \leq \lfloor (d+1)/2 \rfloor$ . Let  $j = \ell$  and  $i = k = \ell'$ . Then the inequalities (14) hold. Thus there exists an integral simplex whose  $\delta$ -vector is  $(1, 0, \dots, 0, 1, 0, \dots, 0, 2, 0, \dots, 0)$ .

On the other hand, we have (12). Then,  $i_2 + i_3 \leq d + 1$  follows from  $i_2 \leq \lfloor (d+1)/2 \rfloor$ .

(3) Assume  $\ell = i_1 = i_2 < i_3 = \ell'$ . Set  $i = \ell'$  and  $j = k = \ell$ . Then it follows from (12) that

$$j + k \geq i, \quad 2j + 1 \leq i + k \leq d + 1 \text{ and } i + j \geq k + 1.$$

Thus, by our result (Table 2, the solution  $d^{(4)}$ ), there exists an integral simplex whose  $\delta$ -vector is  $(1, 0, \dots, 0, 2, 0, \dots, 0, 1, 0, \dots, 0)$ .

On the other hand, if there exists an integral polytope with this  $\delta$ -vector, then (12) holds. In this case, it follows that both inequalities in (13) hold.

(4) Assume  $1 \leq i_1 < i_2 < i_3 \leq d$ . Suppose  $2i_2 \leq i_1 + i_3$  holds. Set  $i = i_3, j = i_2$  and  $k = i_1$ . Then we have  $j + k = i_1 + i_2 \geq i_3 = i$ ,  $2j = 2i_2 \leq i_1 + i_3 = i + k \leq d + 1$  and  $i + j = i_2 + i_3 \geq 2i_2 + 1 \geq 2i_1 + 3 > i_1 + 2 = k + 2$ . Thus, by our result (Table 2, the solution  $d^{(2)}$ ), there exists an integral simplex whose  $\delta$ -vector is  $(1, 0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0)$ .

Suppose  $i_2 + i_3 \leq d + 1$  holds. Set  $i = i_3, j = i_1$  and  $k = i_2$ . Then we have  $j + k = i_1 + i_2 \geq i_3 = i$ ,  $2j = 2i_1 < i_2 + i_3 = i + k \leq d + 1$  and  $i + j = i_1 + i_3 \geq i_1 + i_2 + 1 \geq i_2 + 2 = k + 2$ . Thus, by our result (Table 2, the solution  $d^{(2)}$ ), there exists an integral simplex whose  $\delta$ -vector coincides with  $(\delta_0, \delta_1, \dots, \delta_d)$ .

On the other hand, assume the contrary of (13): both  $2i_2 > i_1 + i_3$  and  $i_2 + i_3 > d + 1$  hold. We claim that there exists no integral polytope  $\mathcal{P}$  with this  $\delta$ -vector. First we want to show that if there exists such a polytope, it must be a simplex. Note that the  $\delta$ -vector satisfies (12). Suppose  $i_1 = 1$ . It then follows from (12) and  $i_2 + i_3 > d + 1$  that  $i_2 = (d+1)/2$  and  $i_3 = (d+3)/2$ . However, this contradicts (3). Therefore  $i_1 > 1$ , and thus  $\delta_1 = 0$ . By an explanation after equation (1),  $\mathcal{P}$  must be a simplex. Now we can apply our characteristic results for simplices.

If we set  $j = i_3$ , then  $2j = 2i_3 > i_1 + i_2 = i + k$ . If we set  $j = i_2$ , then  $2j = 2i_2 > i_1 + i_3 = i + k$ . If we set  $j = i_1$ , then  $i + k = i_2 + i_3 > d + 1$ . In any case there does not exist an Hermite normal form (10) whose  $\delta$ -vector coincides with  $(\delta_0, \delta_1, \dots, \delta_d)$ .

Moreover, since  $i + j + k = i_1 + i_2 + i_3 > i_2 + i_3 > d + 1$ , there does not exist an Hermite normal form (11) with  $\bar{*} = 0$  whose  $\delta$ -vector coincides with  $(\delta_0, \delta_1, \dots, \delta_d)$ .

In addition, if we set  $j = i_3$ , then  $2j = 2i_3 > i_1 + i_2 = i + k$ . If we set  $j = i_2$ , then  $2j = 2i_2 > i_1 + i_3 = i + k$ . If we set  $j = i_1$ , then  $i + k = i_2 + i_3 > d + 1$ . Thus there does not exist an Hermite normal form (11) with  $\bar{*} = 1$  whose  $\delta$ -vector coincides with  $(\delta_0, \delta_1, \dots, \delta_d)$ .  $\square$

**Examples 5.2.** (a) We consider the integer sequence  $(1, 0, 1, 1, 0, 1, 0)$ . Then one has  $i_1 = 2, i_2 = 3, i_3 = 5$  and  $d = 6$ . Since (2) and (3) are satisfied and  $2i_2 \leq i_1 + i_3$  holds, there is an integral polytope whose  $\delta$ -vector coincides with  $(1, 0, 1, 1, 0, 1, 0)$  by Theorem 5.1. In fact, let  $M \in \mathbb{Z}^{6 \times 6}$  be the Hermite normal form (10) with  $(d_1, d_2, d_3) = (0, 1, 4)$  or  $(0, 0, 5)$ . Then we have  $\delta(\mathcal{P}(M)) = (1, 0, 1, 1, 0, 1, 0)$ .

(b) There is no integral polytope with its  $\delta$ -vector  $(1, 0, 1, 0, 1, 1, 0, 0)$  since we have  $2i_2 > i_1 + i_3$  and  $i_2 + i_3 > d + 1$ , although this integer sequence satisfies (2) and (3). (This example is described in [4, Example 1.2] as a counterexample of [4, Theorem 0.1] for the case where  $\sum_{i=0}^d \delta_i = 4$ .) However, there exists an integral polytope with its  $\delta$ -vector  $(1, 0, 1, 0, 1, 1, 0, 0, 0)$  since  $i_2 + i_3 = d + 1$  holds.

**Remark 5.3.** From the above proof, we can see that when  $\sum_{i=0}^d \delta_i = 4$ , all the possible  $\delta$ -vectors can be obtained by simplices. This is also true for all  $\delta$ -vectors with  $\sum_{i=0}^d \delta_i \leq 3$ , from the proof of [4, Theorem 0.1]. However, when  $\sum_{i=0}^d \delta_i = 5$ , the  $\delta$ -vector  $(1, 3, 1)$  cannot be obtained from any simplex, while it is a possible  $\delta$ -vector of a 2-dimensional integral polygon. In fact, suppose that  $(1, 3, 1)$  can be obtained from a simplex. Since  $\min\{i : \delta_i \neq 0, i > 0\} = 1$  and  $\max\{i : \delta_i \neq 0\} = 2$ , one has  $\min\{i : \delta_i \neq 0, i > 0\} = 3 - \max\{i : \delta_i \neq 0\}$ , which implies that the assumption of [5, Theorem 2.3] is satisfied. Thus the  $\delta$ -vector must be shifted symmetric, a contradiction.

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